Introduction Main result and motivation Corollaries and outlook Proof

Asymptotic expansions for fractional stochastic volatility models

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Based on joint works with with Philipp Harms and Antoine Jacquier and with Christian Bayer, Peter Friz, Archil Gulisashvili and Benjamin Stemper

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Implied volatility

- Asset price process: $(S_t = e^{X_t})_{t \geq 0}$, with $X_0 = 0$.
- No dividend, no interest rate.
- Black-Scholes-Merton (BSM) framework:

$$\mathcal{C}_{\mathrm{BS}}(au,k,\sigma) := \mathbb{E}_{0}\left(\mathrm{e}^{X_{ au}} - \mathrm{e}^{k}\right)_{+} = \mathcal{N}\left(d_{+}\right) - \mathrm{e}^{k}\mathcal{N}\left(d_{-}\right),$$

$$d_{\pm} := -\frac{k}{\sigma\sqrt{\tau}} \pm \frac{1}{2}\sigma\sqrt{\tau}.$$

• Spot implied volatility $\sigma_{\tau}(k)$: the unique (non-negative) solution to

$$C_{\text{observed}}(\tau, k) = C_{\text{BS}}(\tau, k, \sigma_{\tau}(k)).$$

• Implied volatility: unit-free measure of option prices.

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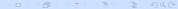
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However the implied volatility is not available in closed form for most models. Its asymptotic behaviour is available via (small/large k, τ) approximations.



Implied volatility $(\sigma_{\tau}(k))$ asymptotics as $|k| \uparrow \infty$, $\tau \downarrow 0$ or $\tau \uparrow \infty$:

- Hagan-Kumar-Lesniewski-Woodward (2003/2015): small-maturity for the SABR model
- Berestycki-Busca-Florent (2004): small- τ using PDE methods for diffusions.
- Henry-Labordère (2009): small-au asymptotics using differential geometry.
- Forde et al.(2012), Jacquier et al.(2012): small/ large au using large deviations.
- Lee (2003), Benaim-Friz (2009), Gulisashvili (2010-2012), Caravenna-Corbetta (2016), De Marco-Jacquier-Hillairet (2013): $|k|\uparrow\infty$.
- ullet Laurence-Gatheral-Hsu-Ouyang-Wang (2012): small-au in local volatility models.
- Fouque et al.(2000-2011): perturbation techniques for slow and fast mean-reverting stochastic volatility models.
- Mijatović-Tankov (2012): small- τ for jump models.
- Bompis-Gobet (2015): asymptotic expansions in the presence of both local and stochastic volatility using Malliavin calculus.

Related works:

- Deuschel-Friz-Jacquier-Violante (CPAM 2014), De Marco-Friz (2014): small-noise expansions using Laplace method on Wiener space (Ben Arous-Bismut approach).
- Baudoin-Ouyang (2010): small-noise expansions in a fractional setting
- Forde-Zhang (2015): large deviations in a fractional stochastic volatility setting
- Guennon-Jacquier-Roome (2015): large deviations in a fractional Heston model



Motivation

- \bullet Classical stochastic volatility models generate a constant short-maturity ATM skew and a large-maturity one proportional to $\tau^{-1};$
- However, short-term data suggests a time decay of the ATM skew proportional to $\tau^{-\alpha}$, $\alpha \in (0,1/2)$.
- One solution: adding volatility factors (risk of over-parameterisation).
 Gatheral's Double Mean-Reverting, Bergomi-Guyon, each factor acting on a specific time horizon.
- In the Lévy case (Tankov, 2010), the situation is different, as $\tau \downarrow 0$:
 - in the pure jump case with $\int_{(-1,1)} |x| \nu(\mathrm{d}x) < \infty$, then $\sigma_{\tau}^2(0) \sim c \tau$;
 - in the (α) stable case, $\sigma_{\tau}^2(0) \sim c \tau^{1-2/\alpha}$ for $\alpha \in (1,2)$;
 - for out-of-the-money options, $\sigma_{ au}^2(k) \sim \frac{k^2}{2 au |\log(au)|}.$

Rough volatility models

 Gatheral-Jaisson-Rosenbaum and Bayer-Gatheral-Friz (2014,1015) proposed a fractional volatility model:

$$dS_t = S_t(\sigma_t dZ_t + \mu_t dt),$$

$$\sigma_t = \exp(X_t),$$
(1)

where

$$X_t = \mu W_t^H - \alpha (X_t - m) dt,$$

for $\mu, \alpha > 0$, $m \in \mathbb{R}$ for a Bm Z and a fBm motion W^H with Hurst parameter H.

- Time series of the Oxford-Man SPX realised variance as well as implied volatility smiles of the SPX suggest that $H \in (0, 1/2)$: short-memory volatility.
- Is not statistically rejected by Ait-Sahalia-Jacod's test (2009) for Itô diffusions.
- Main drawback: loss of Markovianity (H ≠ 1/2) rules out PDE techniques, and Monte Carlo is computationally intensive.

Today's menu

- 1 Introduction Implied volatility
- Main result and motivation Our framework and its scope Examples Density asymptotics
- 3 Corollaries and outlook Short-time expansion Tail expansion Implied volatility asympotics
 - Outlook: Refined expansions and moderate regimes
- 4 Proof Notations Sketch of the proof

$$dX_{t}^{\epsilon} = b_{1}(\epsilon^{\kappa_{1}}, X_{t}^{\epsilon}, Y_{t}^{\epsilon})dt + \epsilon^{\beta} \left(\sigma_{11}(X_{t}^{\epsilon}, Y_{t}^{\epsilon})dW_{t}^{H_{1}} + \sigma_{12}(X_{t}^{\epsilon}, Y_{t}^{\epsilon})dW_{t}^{H_{2}} \right)$$

$$dY_{t}^{\epsilon} = b_{2}(\epsilon^{\kappa_{2}}, X_{t}^{\epsilon}, Y_{t}^{\epsilon})dt + \epsilon^{\beta} \left(\sigma_{21}(X_{t}^{\epsilon}, Y_{t}^{\epsilon})dW_{t}^{H_{1}} + \sigma_{22}(X_{t}^{\epsilon}, Y_{t}^{\epsilon})dW_{t}^{H_{2}} \right).$$

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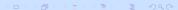
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Assumptions made: $b(\varepsilon,\cdot)\to\sigma_0(\cdot)$ and $x_0^\varepsilon\to x_0$ as $\varepsilon\to 0$, the weak Hörmander condition for $\{\sigma_0,\sigma_1,\ldots,\sigma_d\}$ at x_0 , and vector fields are C^∞ -bounded (can be relaxed).



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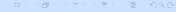
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Our main interest: $H_1 = \frac{1}{2}, H_2 \neq \frac{1}{2}$.



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 σ_1 α -Hölder continuous, $\alpha \in (0,1]$, $\sigma_2 > 0$, (but can be extended to $\sigma_2(\cdot)$ bounded and elliptic) conditions on b_1, b_2 dictated by scaling and existence, and $H \neq 1/2$, particular interest in H < 1/2.

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Remark: The setting of Forde-Zhang '16 is included. With a leap of faith Gatheral-Jaisson-Rosenbaum '14 and Bayer-Friz-Gatheral '15 as well.

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Examples: Consider $(X_0, Y_0) = (0, y_0)$ and

$$dX_{t} = -\frac{Y_{t}^{2}}{2}dt + Y_{t}dW_{t}, \qquad dY_{t} = (a + bY_{t})dt + cdW_{t}^{H}.$$
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Remark: Scaling (4) \Rightarrow short-time (cf. Forde-Zhang (2015)), scaling (5) \Rightarrow tails.



Theorem (Harms-H-Jacquier)

Consider an SDE of the form (2). Then the density of X_T^{ε} admits an expansion

$$f_{\varepsilon}(T,x) = \exp\left(-\frac{\Lambda(x)}{\varepsilon^{2\beta}} + \frac{\Lambda'(x)\widehat{X}_T}{\varepsilon^{\beta}}\right)\varepsilon^{-\min(\kappa_1,\beta)}\left(c_0 + \mathcal{O}(\varepsilon^{\delta(\kappa_1,\beta)})\right), \quad \text{as } \varepsilon \to 0,$$

where

$$\Lambda(x) = \inf \left\{ \frac{1}{2} \|\mathbf{k}\|_{\mathcal{H}_{H}}^{2}, \mathbf{k} \in \mathcal{K}_{x_{0}, y_{0}}^{x} \right\} = \frac{1}{2} \|\mathbf{k}_{0}\|_{\mathcal{H}_{H}}^{2},$$

and

$$\mathrm{d}\widehat{X}_{t} = \left[\partial_{x}b\left(0,\phi_{t}^{k_{0}}\right) + \partial_{x}\sigma\left(\phi_{t}^{h_{0}}\right)\cdot\dot{k}_{0}(t)\right]\widehat{X}_{t}\mathrm{d}t + \partial_{\varepsilon^{\beta}}b\left(0,\phi_{t}^{k_{0}}\right)\mathrm{d}t, \quad \widehat{X}_{0} = \left.\partial_{\varepsilon^{\beta}}x_{0}^{\varepsilon}\right|_{\varepsilon=0},$$

where $\phi^{\mathbf{k}_0}$ denotes the ODE solution of the same SDE (2) replacing $\varepsilon^\beta \mathrm{d} W$ by $\dot{\mathbf{k}}_0$ and \mathbf{x}_0^ε by \mathbf{x}_0 .

Corollary: Varadhan-type asymptotics

Corollary (short-time asymptotics in Stein-Stein) $dY_t = (a + bY_t)dt + cdW_t^H$

Consider the Stein-Stein model (X_t, Y_t) as in (3) with $X_0 = 0$, $Y_0 = y_0 > 0$. Then in a neighbourhood of (x_0, y_0) the density of X_t satisfies the following asymptotic expansion as $t \to 0$

$$f_X(t,x) = \exp\left(\frac{\Lambda(x)}{t^{2H}}\right) t^{-H} \left(\frac{1}{2\pi} + \mathcal{O}(t^{\delta(H,H+1/2)})\right)$$

where

$$\Lambda(x) = \inf \left\{ \frac{1}{2} \left\| \mathbf{k} \right\|_{\mathcal{H}_{H}}^{2}, \mathbf{k} \in \mathcal{K}_{x_{0}, y_{0}}^{x} \right\}.$$

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$$\Lambda(x) = \inf \left\{ \frac{1}{2} \|\mathbf{k}\|_{\mathcal{H}_H}^2, \mathbf{k} \in \mathcal{K}_{x_0, y_0}^x \right\}.$$

Proof: Take T=1, $\varepsilon^2=t$ and consider $(X_t^\epsilon,Y_t^\epsilon):=(\epsilon^{2H-1}X_{\epsilon^2t},Y_{\epsilon^2t})$ with $X_0^\epsilon=0$, $Y_0^\epsilon=y_0>0$. \Rightarrow Short-time scaling:

$$dX_t^{\epsilon} = -\epsilon^{2H+1} \frac{(Y_t^{\epsilon})^2}{2} dt + \epsilon^{2H} Y_t^{\epsilon} dW_t, \qquad dY_t^{\epsilon} = \epsilon^2 (a + bY_t^{\epsilon}) dt + \epsilon^{2H} c dW_t^H, \quad (4)$$

Note that the drift vanishes in the limit $\epsilon \to 0$ and $x_0^\varepsilon = x_0 = 0$. $\Rightarrow (\widehat{X}_t, \widehat{Y}_t) \equiv 0$, so that there is no $1/\epsilon^\beta = 1/t^{\beta/2}$ term in the exponential.

Short-time expansion
Tail expansion
Implied volatility asympotics
Outlook: Refined expansions and moderate regime

Corollary: tail asymptotics

Corollary (tail expansion in Stein-Stein) $dY_t = (a + bY_t)dt + cdW_t^H$

Consider the Stein-Stein model (3) with $X_0=0,\ Y_0=y_0>0.$ Then as $x\to\infty$,

$$f_X(t,x) = \exp\left(-c_1x + c_2x^{1/2}\right) \frac{1}{x^{1/2}} \left(c_0 + \mathcal{O}\left(x^{1/2}\right)\right)$$

where $c_1 := \Lambda(1)$, $c_2 := \widehat{X}_t \Lambda'(1)$.

Note that the expression on the RHS is independent of the Hurst-parameter!

Short-time expansion
Tail expansion
Implied volatility asympotics
Outlook: Refined expansions and moderate regime

Corollary: tail asymptotics

Corollary (tail expansion in Stein-Stein) $dY_t = (a + bY_t)dt + cdW_t^H$

Consider the Stein-Stein model (3) with $X_0 = 0$, $Y_0 = y_0 > 0$. Then as $x \to \infty$,

$$f_X(t,x) = \exp\left(-c_1x + c_2x^{1/2}\right) \frac{1}{x^{1/2}} \left(c_0 + \mathcal{O}\left(x^{1/2}\right)\right)$$

where $c_1 := \Lambda(1)$, $c_2 := \widehat{X}_t \Lambda'(1)$.

Note that the expression on the RHS is independent of the Hurst-parameter!

Proof: Consider $(X_T^{\epsilon}, Y_T^{\epsilon}) := (\epsilon^{2H} X_T, \epsilon^H Y_t)$ with $X_0^{\epsilon} = \epsilon^{2H} X_0$ and $Y_0^{\epsilon} = \epsilon^H Y_0$.

$$dX_t^{\epsilon} = -\frac{(Y_t^{\epsilon})^2}{2}dt + \epsilon^H Y_t^{\epsilon} dW_t, \qquad dY_t^{\epsilon} = (a\epsilon^H + bY_t^{\epsilon})dt + \epsilon^H cdW_t^H, \tag{5}$$

Note that
$$X_t^{\varepsilon} \stackrel{\triangle}{=} \varepsilon^{2H} X_t$$
. $\Rightarrow \mathbb{P}(X_t^{\varepsilon} \geq y) = \mathbb{P}(X_t \geq y/\varepsilon^{2H})$, $\Rightarrow f_X(t,y/\varepsilon^{2H}) = \varepsilon^{2H} f_{\varepsilon}(t,y)$. Take $y=1$, that is $x:=\varepsilon^{-2H}$. By the theorem, $f_{\varepsilon}(t,1) \approx \exp\left(-\frac{\Lambda(1)}{\varepsilon^2 H} + \ldots\right) \frac{1}{\varepsilon^H}$, hence $f_X(t,x) \approx \exp\left(-\frac{\Lambda(1)}{\varepsilon^2 H} + \ldots\right) \varepsilon^H = \exp\left(-\Lambda(1)x + \ldots\right) \frac{1}{\sqrt{1/2}}$.

From density to implied volatility: small-time

Recall the Black-Scholes density expansion:

$$f_{\mathrm{BS}}(t,x) \sim t^{-1/2} \exp\left(-\frac{1}{2t} \left(\frac{x-x_0}{\sigma}\right)^2\right), \quad \text{as } t \to 0, \text{ for any } x \in \mathbb{R}.$$

(We normalise the spot here, so that $x_0 = 0$).

Our theorem (corollary) says that in the Stein-Stein model (3), we have

$$f_{\rm X}(t,x) \sim {\rm cst}\ t^{-H} \exp\left(-\frac{d^2(x_0,y_0;x)}{2t^{2H}}\right),\quad {\rm as}\ t\to 0.$$

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Matching the leading-orders gives

$$\sigma_{\mathrm{BS}}(t,x) \sim rac{|x|}{d(x_0,y_0;x)} t^{H-1/2} \quad ext{as } t
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From density to implied volatility: tails

Recall the Black-Scholes density expansion:

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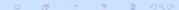
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Matching the leading-orders gives

$$-c_1x + c_2\sqrt{x} \sim -\frac{x^2}{2\sigma^2t} - \frac{x}{4},$$

and we recover Roger Lee's formula independently of the Hurst exponent in (3).



Outlook: Moderate regimes

- Moderate Regimes (in the sense of Friz-Gerhold-Pinter '16) interpolate between out-of-the-money calls with fixed strike $\left(\log\frac{K}{S_0}\right)=k>0$ and at-the-money k=0 calls: Now $k_t=ct^\theta\Rightarrow \mathsf{MOTM}$ (for $0<\theta<\frac{1}{2}$) and AATM (for larger θ)
- Reflects market data: options closer expiry

 strikes closer to the money first observed by Mijatović-Tankov on FX markets
- The moderate regime (MOTM) permits explicit computations for the rate function Λ(k) in terms of the model parameters
 Moderate deviations⇒ Advantage over OTM (large deviations) case where the Λ(k) often related to geodesic distance problems and not explicitly available.
- MOTM expansions naturally involve quantities very familiar to practitioners, notably spot (implied) volatility, implied volatility skew . . .
- In some cases (fractional volatility models) the scaling θ permits a fine-tuning to understand the behavior and derivatives of the energy function.

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Moderate regimes for rough volatility

Rescalings \implies We tacitly agreed to consider $\mathbb{P}\left(X_t \approx t^{1/2-H}x\right)$. Now it is only a small step to consider instead (for some suitable small $\frac{\theta}{0} > 0$)

$$\mathbb{P}\left(X_t\approx t^{1/2-H+\theta}x\right).$$

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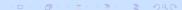
Theorem (Bayer-Friz-Gulisashvili-H-Stemper)

Consider a moderately out-of-the-money call $k_t = x^{1/2-H+\theta}$; $\theta \in (0,H)$ resp. $\theta \in (0,\frac{2H}{3})$. Then as $t \to 0$, the following (non-Markovian extention of Osajima-energy-expansion) holds

$$\log c(k_t, t) \approx \frac{1}{2} \Lambda''(0) \frac{x^2}{t^{2H-2\theta}} + \frac{1}{6} \Lambda'''(0) \frac{x^3}{t^{2H-3\theta}},$$

where we have explicit expressions: $\Lambda''(0)=\frac{1}{\sigma_0})$ and $\Lambda'''(0)=-\rho\frac{6\sigma_0'}{\sigma_0^4}\langle K,1\rangle.$

Here K denotes the Volterra kernel and $\langle K, 1 \rangle := \int_0^1 \int_0^t K(t, s) ds dt$.



Notations

- $\bullet \ \, \mathcal{H} \text{: absolutely continuous paths } [0,T] \to \mathbb{R}^2 \text{ starting at 0 such that } \left\|\dot{h}\right\|_{\mathcal{H}}^2 < \infty.$
- $\mathcal{H}_H := K_H \mathcal{H}$ and $k := K_H h$, where K_H denotes the Volterra kernel.
- For fixed $(x_0, y_0) \in \mathbb{R}^2$, ϕ^k is the (unique) ODE solution to

$$\dot{\phi}_t^{\mathrm{k}} = \sigma_0 \left(\phi_t^{\mathrm{k}} \right) \mathrm{d}t + \sum_{i=1}^m \sigma_i \left(\phi_t^{\mathrm{h}} \right) \mathrm{d}k_t^i, \quad \phi_0^{\mathrm{k}} = (x_0, y_0).$$

- Denote $\psi^k := \Pi_1 \phi^k$ its projection on to the first coordinate X.
- $\mathcal{K}_{\mathrm{a}}:=\left\{\mathbf{k}\in\mathcal{H}_{H}:\psi_{\mathcal{T}}^{\mathbf{k}}=\mathbf{a}\in\mathbb{R}\right\}
 eq\emptyset$ ("by Hörmander condition").
- $\bullet \ \Lambda(\mathrm{a}) := \mathsf{inf} \, \Big\{ \tfrac{1}{2} \, \|\mathrm{k}\|_H^2 : \mathrm{k} \in \mathcal{K}_\mathrm{a} \Big\}.$

$$\label{eq:definition} dX_t = -\epsilon^{2H+1} \frac{1}{2} Y_t^2 dt + \epsilon^{2H} Y_t dW_t, \qquad \qquad dY_t = \epsilon^{2H} dW_t^H,$$

with the same initial condition $X_0 = Y_0 = 0$.

Density:
$$f_{\varepsilon}(T,x) = \exp\left[-\frac{\Lambda(x)}{\varepsilon^{4H}} + \frac{\Lambda'(x)\widehat{X}_T}{\varepsilon^{2H}}\right] \varepsilon^{-2H} \left(c_0 + \mathcal{O}(\varepsilon^{2H})\right).$$

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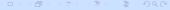
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Proof: Take $x \in \mathbb{R}$ and a C^{∞} -bounded function F such that F(x) = 0.

$$f_{\varepsilon}(T,x)\mathrm{e}^{-F(x)/\varepsilon^{4H}} = \frac{1}{2\pi\varepsilon^{2H}}\int_{\mathbb{R}}\mathbb{E}\left\{\exp\left[\mathrm{i}(\zeta,0)\cdot\left(\frac{X_T^{\varepsilon}-(x,0)}{\varepsilon^{2H}}\right) - \frac{F(X_T^{\varepsilon})}{\varepsilon^{4H}}\right]\right\}\mathrm{d}\zeta.$$

Choose F such that $F(\cdot) + \Lambda_{x_0}(\cdot)$ has a non-degenerate minimum at z. This implies that $k \mapsto F(\phi_T^k(x_0, y_0)) + \frac{1}{2} \|k\|_{\mathcal{H}_H}^2$ has a non-degenerate minimum at $k_0 \in H$.

(For instance
$$F(z) = \lambda |z - x|^2 - [\Lambda_{x_0,y_0}(z) - \Lambda_{x_0,y_0}(x)]$$
 with $\lambda > 0$).



Replace $\varepsilon^{2H}\mathrm{d}B$ $(B:=(W,W^H))$ in the SDE by $\varepsilon^{2H}\mathrm{d}W+\dot{\mathbf{k}}_0$. Call the corresponding Girsanov-transformed process $\widetilde{Z}^{\varepsilon}_t=(\widetilde{X}^{\varepsilon}_t,\widetilde{Y}^{\varepsilon}_t)$:

$$d\widetilde{X}^{\varepsilon} = -\varepsilon^{2H+1} \frac{1}{2} \widetilde{Y}^2 dt + \widetilde{Y}^{\varepsilon} (\varepsilon^{2H} dW_t + (\dot{k}_0)_1), \qquad d\widetilde{Y} = \varepsilon^{2H} dW_t^H + (\dot{k}_0)_2.$$

Girsanov factor

$$\mathcal{G} = \text{exp}\left(-\frac{1}{\epsilon^{2H}}\int_0^T \psi(k_0)_t \text{d}B_t - \frac{1}{2\epsilon^{4H}}\|k_0\|_{\mathcal{H}_H}^2\right).$$

Therefore

$$\begin{split} f(x,T)e^{-F(x)/4\varepsilon^{4H}} &= \frac{1}{2\pi\epsilon^{2H}} \int_{\mathbb{R}} \mathbb{E}\left[e^{\epsilon^{2H}i\zeta(\widetilde{X}_{T}-x)-\varepsilon^{-4H}F(\widetilde{X}_{T})}\mathcal{G}\right] d\zeta \\ &= \frac{1}{2\pi\varepsilon^{2H}} \int_{\mathbb{R}} \mathbb{E}\left[e^{(*)}\right] d\zeta \end{split}$$

where

$$(*) = \varepsilon^{2H} i \zeta(\widetilde{X}_T - x) - \epsilon^{-4H} F(\widetilde{X}_T) - \epsilon^{-2H} \int_0^T \psi(\gamma)_t dB_t - \epsilon^{-4H} \frac{1}{2} \|\gamma\|_{1/2, H}^2.$$

Replace $\varepsilon^{2H}\mathrm{d}B$ $(B:=(W,W^H))$ in the SDE by $\varepsilon^{2H}\mathrm{d}W+\dot{\mathbf{k}}_0$. Call the corresponding Girsanov-transformed process $\widetilde{Z}^\varepsilon_t=(\widetilde{X}^\varepsilon_t,\widetilde{Y}^\varepsilon_t)$:

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By a stochastic Taylor expansion of $\widetilde{Z}^{arepsilon}_t=(\widetilde{X}^{arepsilon}_t,\widetilde{Y}^{arepsilon}_t)$ for $arepsilon^{2H} o 0$,

$$\exp\left(\frac{-F\left(\widetilde{X}_{t}^{\varepsilon}\right)}{\varepsilon^{4H}}\right) = \exp\left[\frac{-1}{\varepsilon^{4H}}\left(F(x) - \varepsilon^{2H}\int_{0}^{T}\psi(\mathbf{k}_{0})_{t}\mathrm{d}B_{t} - \varepsilon^{2H}\widehat{X}_{T}\cdot\Lambda_{x_{0}}'(x) + \mathcal{O}(\varepsilon^{4H})\right)\right]$$

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The rest of the proof follows Ben Arous' proof for X_T^{ε} .

Introduction
Main result and motivation
Corollaries and outlook
Proof

Notations Sketch of the proof

Thank you!